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On Subharmonic Functions which are Bounded Above by Certain Functions

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1. Introduction

Let $X=(x_1,x_2,\text{---},x_k)$ denote a point in the k-dimensional Euclidean space R^k $(k\geqq 1)$ and $\|X\|$ denote the norm of X.

$$\|X\| = \sqrt{x_1^2+x_2^2+\text{---}+x_k^2}.$$

The k-dimensional Lebesgue measure of a set S in R^k is denoted by $|S|$. With a non-negative measurable function f(X) defined on R^m $(m\geqq 1)$, we associate a non-increasing function $\eta=F_f(\xi)$ on the interval $(0,+\infty)$ such that for every $t\geqq 0$ the m-dimensional measure $|S_f(t)|$ of the set

$$S_f(t) = \{X\in R^m |\ f(X)\geqq t\}$$

is equal to the one-dimensional Lebesgue measure of the set

$$\{\xi |\ 0<\xi<+\infty,\ F_f(\xi)\geqq t\}.$$

Such a function $F_f(\xi)$ is obtained by considering the inverse function of $\xi=|S_f(\eta)|$ and is uniquely determined except on a countable set. A non-negative measurable function f(X) on R^m is said to <u>grow slimly</u>, if

$$\text{(1)}\qquad \int_0^\infty \xi^{-(m-1)\ m} \log^+ F_f(\xi)d\xi < +\infty.$$

We note that for a function f(x) defined on R (R^1 is simply denoted by R), (1) is equivalent to the condition

$$\int_{-\infty}^{+\infty} \log^+ f(x)dx < +\infty$$

from the definition of the Lebesgue integral.

Domar [4, Theorem 3] proved the following fact: <u>Let a function</u> $f(X)$ <u>be a slimly growing function on a domain</u> D <u>in</u> R^m <u>and</u> $u(P)$ <u>be subharmonic on the cylinder</u>

$$E = \{P=(X,y) \mid X \in D,\ 0<y<c\},$$

<u>where</u> c <u>is a positive constant, such that</u>

$$u(P) \leqq f(X)$$

<u>for any</u> $P=(X,y)$, $X \in D$, $0<y<c$. <u>Then,</u>

$$u(P) \leqq K$$

<u>on every compact subset of</u> E, <u>where</u> K <u>is a constant independent of</u> $u(P)$.

In this paper, given a slimly growing function $f(X)$ on R^m and some function $h(y)$ on $(0,+\infty)$, we consider an analogous problem to Domar's with respect to a subharmonic function $u(P)$ defined on the $(m+n)$-dimensional Euclidean space R^{m+n} such that

$$u(P) \leqq f(X)h(\|Y\|)$$

for any $P=(X,Y)$, $X \in R^m$, $Y \in R^n$. Using an obtained result, we give a sharpened Phragmén-Lindelöf theorem which extends a result of Deny and Lelong [1], [2] and a result of Brawn [3, Theorem 1].

2. Statements of foundamental results

The proofs of all theorems in this section will be given in the last section. Let $y_0 \geqq 0$ be a constant. A positive non-decreasing function $h(y)$ defined for $(y_0,+\infty)$ is said to grow regularly, if there is a constant $\mu \geqq 1$ such that

$$h(y+1) \leqq \mu h(y)$$

for any $y>y_0$.

The following result is essentially based on Domar's idea in [4].

Theorem 1. Let $f(X)$ be a slimly growing function on R^m and $h(y)$ be a regularly growing function on $(y_0,+\infty)$, $y_0 \geqq 0$, i.e.

$$h(y+1) \leqq \mu h(y)$$

for any $y>y_0$. Suppose that $u(P)$ is a subharmonic function on R^{m+n} such that

$$u(P) \leqq f(X)h(\|Y\|)$$

for any $P=(X,Y)$, $X \in R^m$, $Y \in R^n$, $\|Y\|>y_0$.

Then, there exists a constant K dependent only on $f(X)$ and μ such that

$$u(P) \leqq Kh(\|Y\|)$$

at every $P=(X,Y)$, $X \in R^m$, $Y \in R^n$, $\|Y\|>y_0+2$.

Remark 1. If a function $h(y)$ grows regularly, we can find two positive constants A and B such that

$$h(y) \leqq Ae^{By}$$

to every $y>y_0$. In fact, let y, $y>y_0$, be any number and take a non-negative integer n satisfying

$$n \leqq y-y_0 < n+1.$$

Then,

$$h(y) \leqq h(y_0+(n+1)) \leqq \mu^n h(y_0+1) \leqq \mu^{(y-y_0)} h(y_0+1) = Ae^{By},$$

where

$$A = \mu^{-y_0} h(y_0+1), \qquad B = \log \mu.$$

But, the converse is not always true. Consider the non-decreasing function $h(y)$ on $(0,+\infty)$ defined by

$$h(y) = \int_0^{e^y} \phi(t)dt$$

where

$$\phi(t) = \begin{cases} t & t\in(0,1) \\ (n-1)! & t\in[(n-1)!,n!) \qquad (n=2,3,\text{---}). \end{cases}$$

Then, since $\phi(t) \leqq t$, we have

$$h(y) \leqq 2^{-1}e^{2y}.$$

On the other hand, for a sequence $\{y_n\}$, $y_n=\log n!$ $(n\geqq 2)$,

$$h(1+y_n) = \int_0^{en!} \phi(t)dt > \int_{n!}^{2n!} \phi(t)dt \geqq (n!)^2$$

and

$$nh(y_n) = n\int_0^{n!} \phi(t)dt \leqq n(n-1)!n! = (n!)^2.$$

This shows that $h(y)$ does not grows regularly.

It follows from Remark 1 that $h(y)$ in Theorem 1 must satisfy the growth condition

$$(2) \qquad h(y)=O(e^{By}) \qquad (y\to\infty)$$

for some constant $B>0$. The following Theorem 2 analogous to Otsuka's [6] shows that (2) is almost sharp.

Theorem 2. For any $\varepsilon>0$, there exists a subharmonic function $u_\varepsilon(P)$ on R^{m+n} satisfying the following conditions (i) and (ii);

(i) for a slimly growing function $f_\varepsilon(X)$ on R^m

$$u_\varepsilon(P) \leqq f_\varepsilon(X)e^{\|Y\|^{1+\varepsilon}}$$

at any $P=(X,Y)$, $X\in R^m$, $Y\in R^n$,

(ii) $$\sup_{P=(X,Y),\ X\in R^m,\ Y\in R^n} u_\varepsilon(P)e^{-\|Y\|^{1+\varepsilon}} = +\infty.$$

Question. The function $h(y)=e^{y^{1+\varepsilon}}$ does not grows regularly because it grows quickly. Is it possible to find any result similar to Theorem 2 for a slowly growing function $h(y)$ which does not grow regularly?

The following Theorem 3 shows that the exponent $-(m-1)/m$ of the condition (1) for slim growth of $f(X)$ is best value in Theorem 1.

Theorem 3. There exists a subharmonic function $u(P)$ on R^{m+n} satisfying the following two conditions (i) and (ii);

(i) for a non-negative measurable function $f(X)$ satisfying

$$\int_0^\infty \xi^{-\ell}\log^+F_f(\xi)d\xi < +\infty \quad \text{for any } \ell<(m-1)/m$$

and a regularly growing function $h(y)$ on $(0,+\infty)$,

$$u(P) \leqq f(X)h(\|Y\|)$$

at every $P=(X,Y)$, $X\in R^m$, $Y\in R^n$, $\|Y\|\neq 0$.

(ii) $$\sup_{P=(X,Y),\ X\in R^m,\ Y\in R^n,\ \|Y\|\neq 0} u(P)h(\|Y\|)^{-1} = +\infty.$$

3. Extended Phragmén-Lindelöf theorems

By R^+, we denote the set of positive real numbers. Let G be a domain in R^k $(k \geqq 2)$ and denote the boundary of G by ∂G. When a function u(P) on G is given, we say that u(P) <u>satisfies the Phragmén-Lindelöf boundary condition on</u> ∂G, <u>if</u>

$$\overline{\lim_{P \in G,\ P \to Q}}\ u(P) \leqq 0$$

for every $Q \in \partial G$. When a domain D in R^m and a function $u(P)=u(X,Y)$ on

$$D \times R^n = \{P=(X,Y) \in R^{m+n} \mid X \in D,\ Y \in R^n\}$$

are given, the maximum modulus $M(u,y)$ of $u(P)$ is defined on R^+ by

$$M(u,y) = \sup_{X \in D,\ Y \in R^n,\ \|Y\|=y} u(X,Y),$$

Hardy and Rogosinski [5] proved:

Theorem HR. <u>Let</u> D <u>be an open interval</u> (α,β) <u>and</u> u(Z) <u>be a subharmonic function in the half-strip</u>

$$\Lambda = \{z=x+iy \mid x \in D,\ y \in R^+\}$$

<u>such that</u> u(z) <u>satisfies the Phragmén-Lindelöf boundary condition on</u> $\partial\Lambda$ <u>and</u>

$$\overline{\lim_{y \to \infty}}\ M(u,y)e^{-(\beta-\alpha)^{-1}\pi y} \leqq 0.$$

<u>Then</u>

$$u(z) \leqq 0$$

<u>on</u> Λ.

Deny and Lelong [1], [2] generalized Theorem HR to a function defined on a half-cylinder in the Euclidean space of higher dimension. In the following, a bounded domain in R^m having sufficiently smooth boundary (if m=1, an interval) is called a <u>bounded regular domain</u>. For a given bounded regular domain D, let $\lambda_D>0$ be the first eigenvalue of the boundary value problem with respect to D:

$$\Delta f + \lambda_D f = 0 \quad \text{in } D, \qquad f = 0 \quad \text{on } \partial D$$

where Δ denotes the Laplace operator (if m=1, $\Delta=\frac{d^2}{dx^2}$). If D is an interval (α,β) in R, we easily see

$$\sqrt{\lambda_D} = (\beta-\alpha)^{-1}\pi.$$

Theorem DL. <u>Let</u> D <u>be a bounded regular domain in</u> R^m $(m\geqq 1)$ <u>and</u> u(P) <u>be a subharmonic function in</u> $\Gamma = D\times R^+$ <u>such that</u> u(P) <u>satisfies the Phragmén-Lindelöf boundary condition on</u> $\partial\Gamma$ <u>and</u>

$$\overline{\lim_{y\to\infty}}\, M(u,y)e^{-\sqrt{\lambda_D}y} \leqq 0.$$

<u>Then,</u>

$$u(P)\leqq 0$$

<u>on</u> Γ.

On the other hand, Brawn [3, Theorem 1] generalized Theorem HR to a subharmonic function in the strip $(0,1)\times R^n$ in R^{n+1} $(n\geqq 1)$.

Theorem B. <u>Let</u> u(P) <u>be a subharmonic function in</u>

$$\Omega = (0,1)\times R^n \qquad (n\geqq 1)$$

<u>such that</u> u(P) <u>satisfies the Phragmén-Lindelöf boundary condition on</u> $\partial\Omega$ <u>and</u>

$$\overline{\lim_{y\to\infty}}\, M(u,y)e^{-\pi y}y^{(n-1)/2} \leqq 0.$$

<u>Then</u>

$$u(P) \leqq 0$$

on Ω.

Now, we shall give a generalized form of Theorem DL and Theorem B.

Theorem 4. Let D be a bounded regular domain in R^m $(m \geqq 1)$ and $u(P)$ be a subharmonic function on the domain $\Pi = D \times R^n$ in R^{m+n} such that $u(P)$ satisfies the Phragmén-Lindelöf boundary condition on $\partial\Pi$ and

$$\overline{\lim_{y\to\infty}}\, M(u,y) e^{-\sqrt{\lambda_D}\, y}\, y^{(n-1)/2} \leqq 0.$$

Then,

$$u(P) \leqq 0$$

on Π.

Now, we shall give an extension of Theorem 4.

Theorem 5. Let D be a bounded regular domain in R^m $(m \geqq 1)$ and $u(P)$ be a subharmonic function on the domain $\Pi = D \times R^n$ such that $u(P)$ satisfies the Phragmén-Lindelöf boundary condition on $\partial\Pi$. Suppose that for a slimly growing function $f(X)$ on R^m

$$u(P) \leqq \varepsilon(\|y\|) f(X) e^{\sqrt{\lambda_D}\,\|Y\|} \|Y\|^{(1-n)/2}$$

at every $P=(X,y)$, $X \in D$, $Y \in R^n$, $\|Y\| \neq 0$, where $\varepsilon(t)$ is a function on R^+ satisfying

$$\varepsilon(t) \to 0 \qquad (t \to \infty).$$

Then,

$$u(P) \leqq 0$$

on Π.

Remark 2. If n=1, Theorem 5 extends Theorem DL. If D is (0,1) in R, Theorem 5 extends Theorem B.

The following Theorem 6 shows that the exponent $-(m-1)/m$ in the condition (1) for slim growth of f(X) is best value in Theorem 5.

Theorem 6. There exists an unbounded subharmonic function u(P) on the domain $\Pi_0=D_0\times R^n$ $(n\geqq 1)$,

$$D_0 = \{X\in R^m \mid \|X\| < 2^{-1}\pi\} \quad (m\geqq 1)$$

which satisfies the following conditions (i) and (ii):

(i) u(P) satisfies the Phragmén-Lindelöf boundary condition on $\partial\Pi_0$,

(ii) for a function $\varepsilon(t)$ on R^+ satisfying

$$\varepsilon(t) \to 0 \quad (y\to\infty)$$

and a non-negative measurable function f(X) on R^m satisfying

$$\int_0^\infty \xi^{-\ell}\log^+F_f(\xi)d\xi < +\infty \quad \text{for any } \ell<(m-1)/m,$$

$$u(P) \leqq \varepsilon(\|Y\|)f(X)e^{\sqrt{\lambda_D}\|Y\|}\|Y\|^{(1-n)/2}$$

at every $P=(X,y)$, $X\in D$, $Y\in R^n$, $\|Y\|\neq 0$.

4. Proofs of theorems

By $C_{m+n}(P,r)$, we denote the (m+n)-dimensional ball having a center $P\in R^{m+n}$ and a radius r. To prove Theorem 1, we need the following Lemma which is analogous to Domar's [4, Lemma

2].

Lemma. Let $f(X)$ be a slimly growing function on R^m and $h(y)$ be a regularly growing function on $(y_0,+\infty)$, $y_0 \geqq 0$, i.e.

$$h(y+1) \leqq \mu h(y)$$

for any $y>y_0$. Suppose that $u(P)$ is a subharmonic function on R^{m+n} such that

(3) $$u(P) \leqq f(X)h(\|Y\|)$$

for any $P=(X,Y)$, $X\epsilon R^m$, $Y\epsilon R^n$, $\|Y\|>y_0$. Let Q and λ be positive integers satisfying

$$eA_n A_{m+n}^{-1} Q^{-m} + e^{-\lambda} < \mu^{-1}$$

where

$$A_k = \pi^{k/2}/\Gamma(2^{-1}k+1).$$

If there are an integer ν satisfying

$$0 < Q|S_f(e^{\nu-\lambda})|^{1/m} < 1$$

and a point $P=(X_\nu,Y_\nu)$, $X_\nu \epsilon R^m$, $Y_\nu \epsilon R^n$, $\|Y_\nu\|>y_0+1$ such that

$$u(P_\nu) \geqq e^\nu h(\|Y_\nu\|),$$

then there also exists a point $P_{\nu+1}=(X_{\nu+1},Y_{\nu+1})\epsilon C_{m+n}(P_\nu,r_\nu)$, $X_{\nu+1} \epsilon R^m$, $Y_{\nu+1} \epsilon R^n$,

$$r_\nu = Q|S_f(e^{\nu-\lambda})|^{1/m},$$

such that

$$u(P_{\nu+1}) \geqq e^{\nu+1} h(\|Y_{\nu+1}\|).$$

Proof. First of all, we note that

(4) $$e^\nu h(\|Y_\nu\|) \leqq u(P_\nu) \leqq A_{m+n}^{-1} r_\nu^{-(m+n)} \int_{C_{m+n}(P_\nu,r_\nu)} u(P)dP$$

where dP denotes the $(m+n)$-dimensional volume element (see e.g. Rado [7]).

Now, assume that

$$u(P) < e^{\nu+1}h(\|Y\|)$$

for every $P=(X,Y)\in C_{m+n}(P_\nu, r_\nu)$, $X\in R^m$, $Y\in R^n$. Then,

$$u(P) \leqq e^{\nu+1}h(\|Y_\nu\|+r_\nu) \leqq \mu e^{\nu+1}h(\|Y_\nu\|) \tag{5}$$

for every $P\in C_{m+n}(P_\nu, r_\nu)$. If we put

$$S = C_{m+n}(P_\nu, r_\nu) \cap \{S_f(e^{\nu-\lambda})\times R^n\},$$

we have

$$|S| \leqq A_n r_\nu^n |S_f(e^{\nu-\lambda})| = A_n Q^{-m} r_\nu^{m+n} \tag{6}$$

and

$$u(P) \leqq e^{\nu-\lambda}h(\|Y\|) \leqq e^{\nu-\lambda}h(\|Y_\nu\|+r_\nu) \leqq \mu e^{\nu-\lambda}h(\|Y_\nu\|) \tag{7}$$

for every $P=(X,Y)\in C_{m+n}(P_\nu, r_\nu)-S$, from (3). Thus, we obtain

$$A_{m+n}^{-1} r_\nu^{-(m+n)} \int_{C_{m+n}(P_\nu, r_\nu)} u(P)dP =$$

$$A_{m+n}^{-1} r_\nu^{-(m+n)} \int_S u(P)dP + A_{m+n}^{-1} r_\nu^{-(m+n)} \int_{C_{m+n}(P_\nu, r_\nu)-S} u(P)dP$$

$$\leqq A_{m+n}^{-1} r_\nu^{-(m+n)} \mu e^{\nu+1} h(\|Y_\nu\|)|S|$$

$$+ A_{m+n}^{-1} r_\nu^{-(m+n)} \mu e^{\nu-\lambda} h(\|Y_\nu\|)|C_{m+n}(P_\nu, r_\nu)-S|$$

$$\leqq (eA_{m+n}^{-1}A_n Q^{-m} + e^{-\lambda})\mu e^{\nu} h(\|Y_\nu\|) < e^{\nu}h(\|Y_\nu\|),$$

from (5), (6) and (7). But, this contradicts (4).

Proof of Theorem 1. If we put

$$a_k = |S_f(e^k)|,$$

then

$$\sum_{k=1}^{\infty} |S_f(e^k)|^{1/m} = m\sum_{k=1}^{\infty} \int_0^{a_k} \xi^{-(m-1)/m} d\xi = m\sum_{k=1}^{\infty} \int_{a_{k+1}}^{a_k} k\xi^{-(m-1)/m} d\xi$$

$$\leqq m\sum_{k=1}^{\infty} \int_{a_{k+1}}^{a_k} \xi^{-(m-1)/m} \log^+ F_f(\xi)d\xi \leqq m\int_0^{\infty} \xi^{-(m-1)/m} \log^+ F_f(\xi)d\xi .$$

Hence, we see that the series

$$\sum_{k=1}^{\infty} |S_f(e^k)|^{1/m}$$

converges.

Now, we shall prove by dividing into two cases.

(Case 1) We consider the case where

$$|S_f(e^k)| > 0$$

for any positive integer k. For the integer Q and λ (which are dependent on μ) chosen in Lemma, take a sufficiently large integer ν_0 such that

$$(8) \qquad \sum_{\nu=\nu_0}^{\infty} |S_f(e^{\nu-\lambda})|^{1/m} < Q^{-1}.$$

Here, we remark that ν_0 depends on $f(X)$ and μ.

Now, assume that there is a point $P_{\nu_0}=(X_{\nu_0}, Y_{\nu_0})$, $X_{\nu_0} \in R^m$, $Y_{\nu_0} \in R^n$, $\|Y_{\nu_0}\| > y_0+2$, such that

$$u(P_{\nu_0}) \geqq e^{\nu_0} h(\|Y_{\nu_0}\|).$$

If we put

$$r_{\nu_0} = Q|S_f(e^{\nu_0-\lambda})|^{1/m}$$

and apply Lemma, we can find a point

$$P_{\nu_0+1}=(X_{\nu_0+1}, Y_{\nu_0+1}) \in C_{m+n}(P_{\nu_0}, r_{\nu_0}),\ X_{\nu_0+1} \in R^m,\ Y_{\nu_0+1} \in R^n,$$

such that

$$u(P_{\nu_0+1}) \geqq e^{\nu_0+1} h(\|Y_{\nu_0+1}\|).$$

Here, if we see

$$\|Y_{\nu_0+1}\| \geqq \|Y_{\nu_0}\| - r_{\nu_0} > y_0+1$$

and put

$$r_{\nu_0+1} = Q|S_f(e^{\nu_0+1-\lambda})|^{1/m},$$

we can also apply Lemma and find a point

$$P_{\nu_0+2}=(X_{\nu_0+2}, Y_{\nu_0+2}) \in C_{m+n}(P_{\nu_0+1}, r_{\nu_0+1}),\ X_{\nu_0+2} \in R^m,\ Y_{\nu_0+2} \in R^n,$$

such that

$$u(P_{\nu_0+2}) \geqq e^{\nu_0+2} h(\|Y_{\nu_0+2}\|).$$

Here, we see

$$\|P_{\nu_0+2} - P_{\nu_0}\| \leqq r_{\nu_0} + r_{\nu_0+1}$$

$$= Q(|S_f(e^{\nu_0-\lambda})|^{1/m} + |S_f(e^{\nu_0+1-\lambda})|^{1/m}) < 1$$

from (8), which gives

$$\|Y_{\nu_0+2}\| \geqq \|Y_{\nu_0}\|-1 > y_0+1.$$

Thus, if we continue this process, we can obtain a sequence of points

$$\{P_{\nu_0+i}\}_{i=0}^{\infty},\ P_{\nu_0+i}=(X_{\nu_0+i}, Y_{\nu_0+i}),\ X_{\nu_0+i}\in R^m,\ Y_{\nu_0+i}\in R^n,$$

such that

$$\|P_{\nu_0+i} - P_{\nu_0}\| < 1$$

and

$$u(P_{\nu_0+i}) \geqq e^{\nu_0+i} h(\|Y_{\nu_0+i}\|) \geqq e^{\nu_0+i} h(y_0+1) \to \infty \quad (i\to\infty).$$

These show that $u(P)$ is unbounded above on $C_{m+n}(P_{\nu_0}, 1)$. This contradicts the boundedness of $u(P)$ on $C_{m+n}(P_{\nu_0}, 1)$.

Thus, if we put $K=e^{\nu_0}$, we have

$$u(P) \leqq Kh(\|Y\|)$$

for any $P=(X,Y)$, $X\in R^m$, $Y\in R^n$, $\|Y\|>y_0+2$.

(Case 2) Suppose that Case 1 does not happen i.e., there is a k_0 such that

$$|S_f(e^{k_0})| = 0.$$

Take any $P'=(X',Y')$, $X'\in R^m$, $Y'\in R^n$, $\|Y'\|>y_0$, and a positive number δ', $\delta'<\min(1, \|Y'\|-y_0)$.

If we put

$$S' = C_{m+n}(P', \delta') \cap \{(X,Y)|\ X\in R^m,\ Y\in R^n,\ X\in S_f(e^{k_0})\},$$

we have

(9) $$|S'| \leqq |S_f(e^{k_0})| A_n \delta'^n = 0$$

and

(10) $$u(P) \leqq h(\|Y\|) f(X) < h(\|Y'\| + \delta') e^{k_0} \leqq \mu h(\|Y'\|) e^{k_0}$$

for any $P=(X,Y)\in C_{m+n}(P', \delta')-S'$, $X\in R^m$, $Y\in R^n$. Hence, if we denote by M' the maximum of $u(P)$ on $C_{m+n}(P', \delta')$,

$$u(P') \leqq A_{m+n}^{-1} \delta'^{-(m+n)} \int_{C_{m+n}(P', \delta')} u(P)dP$$

$$= A_{m+n}^{-1} \delta'^{-(m+n)} \int_{S'} u(P)dP + A_{m+n}^{-1} \delta'^{-(m+n)} \int_{C_{m+n}(P', \delta')-S'} u(P)dP$$

$$\leqq M' A_{m+n}^{-1} \delta'^{-(m+n)} |S'| + \mu h(\|Y'\|) e^{k_0} = \mu h(\|Y'\|) e^{k_0}$$

from (9) and (10).

Thus, putting $\mu e^{k_0} = K$, we have

$$u(P) \leqq Kh(\|Y\|)$$

for any $P=(X,Y)$, $X\in R^m$, $Y\in R^n$, $\|Y\|>y_0$.

Proof of Theorem 2. Given any $\varepsilon>0$, consider the function $u_\varepsilon^*(P)$ on R^{m+n} defined by

$$u_\varepsilon^*(P) = \begin{cases} \|Y\|^\varepsilon (\cos\|X\|) \exp(\|Y\|^{1+\varepsilon} - \|X\|^2 \|Y\|^\varepsilon) & \\ & \text{on } \{P=(X,Y) \mid X\in R^m, \|X\|<2^{-1}\pi, Y\in R^n\} \\ 0 & \text{elsewhere.} \end{cases}$$

If we write $\|X\|=x$, $\|Y\|=y$ and

$$g(x,y) = \exp(y^{1+\varepsilon} - x^2 y^\varepsilon)$$

for simplicity, we have

$$\Delta u_\varepsilon^* = \frac{\partial^2 u_\varepsilon^*}{\partial x^2} + \frac{m-1}{x} \frac{\partial u_\varepsilon^*}{\partial x} + \frac{n-1}{y} \frac{\partial u_\varepsilon^*}{\partial y} + \frac{\partial^2 u_\varepsilon^*}{\partial y^2}$$

$$\geqq g(x,y)[y^{3\varepsilon}\{(1+\varepsilon)^2 - o(1)\}\cos x + y^{2\varepsilon}\{4 - x^{-2}(m-1)y^{-\varepsilon}\}] x \sin x$$

$$\geqq \begin{cases} g(x,y)[y^{3\varepsilon}\{(1+\varepsilon)^2-o(1)\}\cos x + y^{2\varepsilon}\{2^{-1}\sqrt{2}-o(1)\}] \\ \hfill (4^{-1}\pi \leqq x \leqq 2^{-1}\pi,\ y\to\infty) \\ g(x,y)y^{3\varepsilon}\{2^{-1}\sqrt{2}(1+\varepsilon)^2-o(1)-(m-1)y^{-2\varepsilon}x^{-1}\sin x\} \\ \quad \geqq g(x,y)y^{3\varepsilon}\{2^{-1}\sqrt{2}(1+\varepsilon)^2-o(1)\} \qquad (0<x\leqq 4^{-1}\pi,\ y\to\infty) \end{cases}$$

by an elementary computation. This shows that $u^*_\varepsilon(P)$ is subharmonic on $\{P=(X,Y) \mid X\in R^m,\ Y\in R^n,\ \|Y\|>a\}$ for a sufficiently large a. Here, choose a constant M_ε so that

$$u^*_\varepsilon(P) \leqq M_\varepsilon \quad \text{on } \{P=(X,Y) \mid X\in R^m,\ Y\in R^n,\ \|Y\|<2a\}$$

and define $u_\varepsilon(P)$ by

$$u_\varepsilon(P) = \max\{u^*_\varepsilon(P),\ M_\varepsilon\}.$$

Then, $u_\varepsilon(P)$ is a subharmonic function on R^{m+n} which is requested in Theorem 2.

First, for the function

$$(11) \qquad f_\varepsilon(X) = \max\{\|X\|^{-2},\ M_\varepsilon\}$$

on R^m, we shall show the inequality of (i) in Theorem 2.

Set

$$\psi(x,y) = x^{-2}-y^\varepsilon\exp(-x^2y^\varepsilon)$$

for (x,y), $x\in R^+$, $y\in R^+$. Then, we have

$$\frac{\partial\psi}{\partial y} = (-\varepsilon y^{\varepsilon-1} + \varepsilon x^2y^{2\varepsilon-1})\exp(-x^2y^\varepsilon)$$

which vanishes at $y_0=x^{-2/\varepsilon}$. Further,

$$\psi(x,y_0) = x^{-2}-e^{-1}x^{-2} > 0$$

and

$$\psi(x,y) \to x^{-2} \quad \text{as } y\to 0 \text{ and } y\to\infty.$$

Hence,

$$\psi(x,y) > 0, \text{ i.e. } x^{-2} > y^\varepsilon\exp(-x^2y^\varepsilon)$$

on $R^+\times R^+$. From this fact, the required inequality immediately follows.

Here, it is easy to see that $f_\varepsilon(X)$ in (11) is a slimly growing function on R^m, because

$$F_{f_\varepsilon}(\xi) = (A_m \xi^{-1})^{2/m}$$

at every $\xi < A_m M_\varepsilon^{-m/2}$.

To obtain (ii) in Theorem 2, observe

$$u_\varepsilon(0,Y)e^{-\|Y\|^{1+\varepsilon}} \to +\infty$$

uniformly as $\|Y\| \to +\infty$.

Proof of Theorem 3. Put

$$V(P) = \exp(e^{\|Y\|}\cos\|X\|)\cos(e^{\|Y\|}\sin\|X\|)$$

for any $P=(X,Y)$, $X \in R^m$, $Y \in R^n$ and consider the function

$$U^*(P) = \{V(P)\}^{2m-1}$$

defined on $R^m \times R^n$. If we write $\|X\|=x$ and $\|Y\|=y$, we have

$$\Delta U^* = (2m-1)V^{2m-2}[(2m-2)\{(\frac{\partial V}{\partial x})^2 + (\frac{\partial V}{\partial y})^2\} + V\Delta V]$$

$$= (2m-1)V^{2m-2}\exp(y+2e^y\cos x)g(x,y)$$

where

$$g(x,y) = (2m-2)e^y$$
$$+ \cos(e^y\sin x)\{\frac{n-1}{y}\cos(x+e^y\sin x) - \frac{m-1}{x}\sin(x+e^y\sin x)\}.$$

Here, if

$$0 < x < \pi/2 \quad \text{and} \quad x + e^y\sin x < \pi/2,$$

we see that

$$\sin(x+e^y\sin x) \leqq x+e^y\sin x \leqq x(1+e^y)$$

and hence

$$g(x,y) \geqq (m-1)(e^y-1) + \frac{n-1}{y}\cos(e^y\sin x)\cos(x+e^y\sin x) \geqq 0.$$

Hence, we have

$$\Delta U^* \geqq 0$$

for any $P=(X,Y)$, $X \in R^m$, $Y \in R^n$, $\|X\| < \pi/2$, $\|X\| + e^{\|Y\|}\sin\|X\| < \pi/2$.

Let

$$D_0 = \{X \in R^m \mid \|X\| < \pi/2\}$$

and

$$S = \{(X,Y)\epsilon R^{m+n} \mid X\epsilon D_0,\ Y\epsilon R^n,\ \sin\|X\|<2^{-1}\pi e^{-\|Y\|},\ \|Y\|>y_0\},$$

where $y_0=\log 2^{-1}\pi$. Choose a positive constant M such that

$$U^*(P) \leqq M$$

on $D_0\times\{Y\epsilon R^n \mid \|Y\|<2y_0\}$ and define the function $u(P)$ on R^{m+n} by

$$u(P) = \begin{cases} M^{-1}\max\{U^*(P),\ M\} & \text{on } S \\ 1 & \text{elsewhere,} \end{cases}$$

which is a subharmonic function requested in Theorem 3.

Now, if we define $f(X)$ on R^m by

$$(12) \qquad f(X) = \sup_{Y\epsilon R^n} u(X,Y)$$

and $h(y)$ on R^+ by

$$h(y) \equiv 1,$$

we have the inequality of (i) in Theorem 3. Here, it is evident that $h(y)$ is a regularly growing function on R^+. Hence, we shall show that

$$(13) \qquad \int_0^\infty \xi^{-\ell}\log^+F_f(\xi)d\xi < +\infty \quad \text{for any } \ell,\ \ell<(m-1)/m.$$

Put

$$v(x,y) = \exp(e^y\cos x)\cos(e^y\sin x)$$

for $x\epsilon R$, $y\epsilon R$, $y>y_0=\log 2^{-1}\pi$. Then, for any fixed y, $v(x,y)$ increases from 0 to $\exp(e^y)$ as x decreases from $\sin^{-1}(2^{-1}\pi e^{-y})$ to 0. This fact gives that

$$u(P) > t$$

on the domain which is surrounded by the set

$$\{P\epsilon D_0\times R^n \mid P\epsilon S,\ V(P)=t\}$$

for a sufficiently large t. For a given t, consider the curve

$$L = \{(x,y)\epsilon R^2 \mid v(x,y)=t,\ 0\leqq x<\pi/2\}$$

in the plane and put

$$x_0 = \max_{(x,y)\in L} x.$$

Since

$$\frac{dy}{dx} = -\tan(x+e^y \sin x)$$

along L, we have

$$x_0 + e^y \sin x_0 = \pi/2.$$

Hence, x_0 satisfies

$$\exp\{(2^{-1}\pi - x_0)\cot x_0\}\sin x_0 = t.$$

Since

$$|S_f(t)| = A_m x_0^m$$

for a sufficiently large t from the definition (12) of f(X), we have

$$F_f(\xi) = \exp[\{2^{-1}\pi-(A_m^{-1}\xi)^{1/m}\}\cot\{(A_m^{-1}\xi)^{1/m}\}]\sin\{(A_m^{-1}\xi)^{1/m}\}.$$

Thus, for a suffiently small $\xi>0$,

$$K_1\xi^{-1/m} \leqq \log F_f(\xi) \leqq K_2\xi^{-1/m}$$

where K_1 and K_2 are two positive constants. This gives (13).

The conclusion (ii) in Theorem 3 immediately follows for these u(P) and h(y) from the fact

$$u(0,Y) = M^{-1}\exp\{(2m-1)e^{\|Y\|}\}$$

at any $Y\in R^n$ having sufficiently large $\|Y\|$.

Proof of Theorem 4. This theorem is proved by following both methods used to prove Theorem DL and Theorem B. For a given bounded regular domain D, we denote the positive eigenfunction corresponding to the eigenvalue λ_D by $f_D(X)$ and define $h_D(P)$ on

$$D\times R^n = \{P=(X,y)|\ X\in D,\ Y\in R^n\}$$

by

$$h_D(P) = f_D(X)\|Y\|^{1-n/2} I_{n/2-1}(\sqrt{\lambda_D}\|Y\|),$$

where $I_{n/2-1}(y)$ is the Bessel function of the third kind, of order n/2-1 (see e.g. Watson [8, p.77]). It is easy to see that $h_D(P)$ is harmonic on $D\times R^n$. We also remark that

$$I_{n/2-1}(y) = (2\pi y)^{-1/2} e^y (1+o(1)) \qquad (y\to+\infty)$$

(see Watson [8, p.203]).

Now, consider the subharmonic function $u_1(P)$ on Π defined by

$$u_1(P) = u(P) - \eta_1 h_D(P) \qquad (\eta_1 > 0).$$

Take a closed ball $B \subset D$ and choose a positive constant ε_1 such that

$$f_D(X) \geqq \varepsilon_1 \quad \text{on } B.$$

If we choose a positive constant y_1 such that

$$M(u,y) < 2^{-1}\varepsilon_1\eta_1 C_D e^{\sqrt{\lambda_D}y} y^{(1-n)/2}$$

for any $y \geqq y_1$, where

$$C_D = (2\pi\sqrt{\lambda_D})^{-1/2},$$

we see that

$$u_1(P) \leqq \varepsilon_1\eta_1 C_D\{-2^{-1}-o(1)\} e^{\sqrt{\lambda_D}\|Y\|}\|Y\|^{(1-n)/2}$$

for any $P=(X,Y)$, $X\in B$, $\|Y\| \geqq y_1$. Hence, there are a value M and a point $P_0 \in B\times R^n$ such that

(14) $$u_1(P_0) = M \quad \text{and} \quad u_1(P) \leqq M \quad \text{on } B.$$

Next, take a bounded regular domain D^*, D^* R^m such that

$$\partial(D-B) \cup (D-B) \subset D^* \quad \text{and} \quad \lambda_D < \lambda_{D^*} < \lambda_{D-B}.$$

Consider the subharmonic function $u_2(P)$ on $(D^*-B)\times R^n$ defined by

$$u_2(P) = u_1(P) - \eta_2 h_{D^*}(P) \qquad (\eta_2 > 0).$$

If we take a positive number ε_2 such that

$$f_{D^*}(X) \geqq \varepsilon_2 \quad \text{on } \partial(D-B)\cup(D-B)$$

and a number y_2 such that

$$M(u,y) < \varepsilon_2 \eta_2 c_{D*} e^{\sqrt{\lambda_D} y} y^{(1-n)/2}$$

for any $y \geqq y_2$, we have that

$$u_2(P) \leqq u(P) - \eta_2 h_{D*}(P)$$
$$\leqq \varepsilon_2 \eta_2 c_{D*} \{ e^{(\sqrt{\lambda_D} - \sqrt{\lambda_{D*}})\|Y\|} - (1+o(1)) \} e^{\sqrt{\lambda_{D*}}\|Y\|} \|Y\|^{(1-n)/2}$$

for any $P=(X,Y) \in D \times R^n - B$, $\|Y\| \geqq y_2$. Hence, with (14) the maximal principle gives that

$$u_2(P) \leqq \max(0,M) \quad \text{on } D-B.$$

Thus, we have that

$$u_1(P) \leqq \max(0,M) \quad \text{on } D-B,$$

because η_2 is chosen arbitrarily small. Further, we have from (14) that

$$u_1(P) \leqq \max(0,M) \quad \text{on } D.$$

By (14) and the maximal principle, this gives that

$$M \leqq 0 \quad \text{and hence} \quad u_1(P) \leqq 0 \quad \text{on } D.$$

As $\eta_1 \to 0$, we can conclude that

$$u(P) \leqq 0 \quad \text{on } D.$$

Proof of Theorem 5. For each positive integer m, take a number t_m such that

$$\varepsilon(t) \leqq 1/m$$

for every $t \geqq t_m$. Then

$$u(P) \leqq f(X)\{ m^{-1} e^{\sqrt{\lambda_D}\|Y\|} \|Y\|^{(1-n)/2} \}$$

at every $P=(X,Y)$, $X \in D$, $Y \in R^n$, $\|Y\| \geqq t_m$. If we put

$$h_m(y) = m^{-1} e^{\sqrt{\lambda_D} y} y^{(1-n)/2}$$

it is easy to see that $h_m(y)$ is a regularly growing function on $(t_m,+\infty)$, i.e.

$$h_m(y+1) \leqq e^{\sqrt{\lambda_D}} h_m(y)$$

for every $y>t_m$. Hence, if we also put $u(P)=0$ on $R^{m+n}-\Pi$ and apply Theorem 1, there exists a constant K independent of m such that

$$u(P) \leqq Kh_m(\|Y\|) = m^{-1}Ke^{\sqrt{\lambda_D}\|Y\|}\|Y\|^{(1-n)/2}$$

for every $P=(X,Y)$, $X\epsilon D$, $Y\epsilon R^n$, $\|Y\|>t_m+2$. This gives that

$$\overline{\lim_{y\to\infty}}\, M(u,y)e^{-\sqrt{\lambda_D}y}y^{(n-1)/2} \leqq 0.$$

Hence, from Theorem 4, the conclusion follows.

Proof of Theorem 6. For the function $u(P)$ and the constant taken in the proof of Theorem 3, consider the function $u(P)-1$ on $\Pi_0=D_0\times R^n$. When we represent this function by $u(P)$ again, we shall show that $u(P)$ is the subharmonic function requested in Theorem 6. The statement (i) in Theorem 6 is evident. To prove the statement (ii) in Theorem 6, define $f(X)$ on R^m by

$$f(X) = \begin{cases} \sup\limits_{Y\epsilon R^n} u(X,Y) & \text{on } D \\ 0 & \text{elsewhere} \end{cases}$$

and $\varepsilon(t)$ on R^+ by

$$\varepsilon(t) = e^{-\sqrt{\lambda_D}t}t^{(n-1)/2}.$$

Then,

$$u(P) \leqq \varepsilon(\|Y\|)f(X)e^{-\sqrt{\lambda_D}\|Y\|}\|Y\|^{(1-n)\ 2}$$

for any $P=(X,Y)$, $X\epsilon D_0$, $Y\epsilon R^n$, $\|Y\|\neq 0$. The finiteness of the integral

$$\int_0^\infty \xi^{-\ell}\log^+ F_f(\xi)d\xi$$

for any $\ell < (m-1)/m$ follows immediately from the proof of Theorem 3.

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